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THE JOURNAL

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# Indian Mathematical Society.

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## PROGRESS REPORT.

1. The following gentlemen have been elected members of the Society:

1. Mr. K. S. Appaswami Aiyar B.A., L. T., Sub-Assistant Inspector of Schools, Coimbatore.
2. Mr. T. R. Panchapagesan M.A.,L.T., Assistant Professor, Presidency College, Madras.
3. Mr. K. Amrita Row, B.A.,L.T., Assistant Professor, Presidency College, Madras.
4. Mr. V. Gopalayya B.C.E., (Assistant Engineer, P. W. D. Madras) special Engineer, T. R. S. Works, Parlakimidi.

2. The Committee feel great pleasure in announcing to the General Body that Mr. Dadabhai Naoroji, the Grand Old Man of India, and the first Indian appointed as Professor of Mathematics, has honoured the Society by accepting the offer of an Honorary Membership of the Society. The Committee feel sure that the General Body cannot but fail to appreciate the honour done to the Society by Mr. Dadabhai Naoroji.

3. The Committee also report for information that owing to the absence on furlough of Professor A. C. L. Wilkinson M. A., F.R.A.S., Mr. M. P. Khareghat I.C.S., (retired) has been taken on the Committee for the remaining part of this year.

4. The following book has been purchased for the Library
*Theory of Functions of Real Variables*—by James Pierpont (Ginn & Co, New York) Price 20s.

5. *The Calcutta University Calendar for 1911—Parts I, II, III,* has been presented to our Library.

POONA,
*1st October 1911.*

D. D. KAPADIA,
*Hon. Joint Secretary.*

# An Introduction to the theory of moving axes with application to curves in space and curves on surfaces.

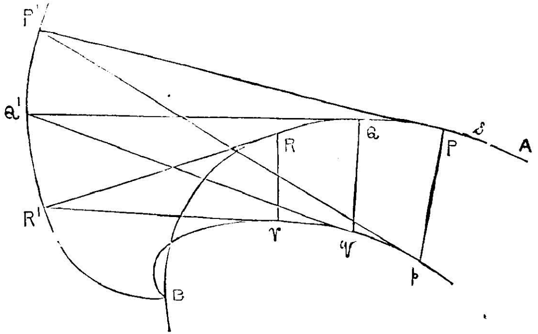
By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page 111*).

§ 16. *On Involutes and Evolutes.*

A curve in space has an infinite number both of involutes and evolutes.

Consider a curve PQR... ; along the tangents at P,Q,R...take lengths PP′, QQ′, RR′...equal to the arcs PB, QB, RB..., then the curve P′Q′R′... is an *involute* of PQR... By varying the position of B, we obtain an infinite number of involutes. We shall show that P,Q,R... lie in the normal planes of P′Q′R′...and thus PQR...is an *evolute* of P′Q′R′... If other normals P′*p*, Q′*q*, R′*r*...are drawn making a constant angle with P′P, Q′Q, R′R, we shall show that these also envelope a curve, and thus *pqr*...is also an evolute of P′Q′R′... Considering any two curves related as PQR..., P′Q′R′..., we shall call PQR...the evolute and P′Q′R′...the involute.




Take moving axes at P. Then $\xi = ds$, $\eta = o$, $\zeta = o$, $\partial_1 = dT$, $\partial_2 = o$, $\partial_3 = d\varepsilon$. Let $s$ be the length of the arc AP and $c$ the length of the arc PB. Denote the elements of arc, contingence angle, and torsion angle at P′ by $\Delta s$, $\Delta\varepsilon$, $\Delta T$. The coordinates of P′ are $(-s+c,\ o,\ o)$; the consecutive point Q′ is therefore

$$(ds-s-ds+c),\ (-s+c)\,d\varepsilon,\ o$$

Thus the direction cosines of the tangent at P′ are $(o, 1, o)$, whence, *the tangent at any point of the involute is parallel to the principal normal at the corresponding point of the evolute.*

The direction cosines of the consecutive tangent are $(-d\epsilon, 1, dT)$. Therefore for the radius of curvature at P′, we have.

$$\Delta\epsilon^2 = d\epsilon^2 + dT^2\ ;\ \ \Delta s^2 = (c-s)^2\, d\epsilon^2$$

$$\therefore \rho'^2 = \sigma^2 (c-s)^2/(\sigma^2+\rho^2)$$

The binormal at P′ being perpendicular to $(o, 1, o)$, and $(-d\epsilon, 1, dT)$ will have direction cosines proportional to $(dT, o, d\epsilon)$. The direction cosines of the binormal are therefore

$$(\rho/\sqrt{\rho^2+\sigma^2},\ o,\ \sigma/\sqrt{\rho^2+\sigma^2})$$

Hence, *the binormal at any point of the involute is parallel to the rectifying line at the corresponding point of the evolute.*

Denoting the direction cosines of the binormal by $(\cos\gamma, o, \sin\gamma)$ where $\tan\gamma = \sigma/\rho$, we obtain for the torsion

$$\Delta T^2 = (-\sin\gamma\, d\gamma)^2 + (-\sin\gamma\, dT + \cos\gamma\, d\epsilon)^2 + (\cos\gamma\, d\gamma)^2\ ;$$

whence, $\qquad \Delta T = d\gamma$

$$\therefore \qquad 1/\sigma' = \frac{\rho}{c-s}\,\frac{d}{ds}\left(\tan^{-1}\frac{\sigma}{\rho}\right)$$

The condition that the involute may be a plane curve is given by $\gamma = \text{constant}$ ; or, $\sigma/\rho = \text{constant}$. Thus, *if the involutes of a curve are plane curves, the curve is a geodesic on a cylinder.*

§ 17. *To find the evolutes of a given curve.*

The normals must be tangents to some curve. Consider a normal making an angle $\gamma$ with the principal normal, and take a distance $\xi$ along it. The coordinates of the point obtained are

$$(o,\ \xi\cos\gamma,\ \xi\sin\gamma)\ ;$$

the consecutive point is

$$ds - \xi\cos\gamma\, d\epsilon,\ \xi\cos\gamma + d(\xi\cos\gamma) - \xi\sin\gamma\, dT,\ \xi\sin\gamma + d(\xi\sin\gamma) + \xi\cos\gamma\, dT.$$

Thus, the direction cosines of the tangent are

$$ds - \xi\cos\gamma\, d\epsilon,\ d(\xi\cos\gamma) - \xi\sin\gamma\, dT,\ d(\xi\sin\gamma) + \xi\cos\gamma\, dT.$$

and, if these are identical with $(o, \cos\gamma, \sin\gamma)$, we have,

$$\rho = \xi\cos\gamma$$

$$\frac{d(\xi\cos\gamma) - \xi\sin\gamma\, dT}{\cos\gamma} = \frac{d(\xi\sin\gamma) + \xi\cos\gamma\, dT}{\sin\gamma}$$

or, $$\frac{-\xi\sin\gamma\,(d\gamma + dT)}{\cos\gamma} = \frac{\xi\cos\gamma\,(d\gamma + dT)}{\sin\gamma},$$

whence, $\qquad d\gamma + dT = o$, and $\gamma = c - T$, where $c$ is constant.

Thus, any point on an evolute is given by

$$\{ o , \rho , \rho \tan (c-T) \}$$

The following theorems are now clear :

1. *The evolutes of a curve in space are the edges of regression of those developable surfaces whose generators are normals to the curve.*
2. *The evolutes all lie on the polar developable of the curve.*
3. *Considering any two evolutes $PQR\ldots$, $pqr\ldots$ of a curve $P'Q'R'\ldots$ the angle $PP'p$ is constant.*
4. *If through every point of an orthogonal trajectory of the generators of a developable surface, straight lines are drawn in the normal planes of the points on the trajectory making a constant angle with the generators, these straight lines generate a developable surface.*

These theorems are given in Kommerell, "Raumkurven and Flaçhen", I Band.

All the evolutes of a curve are geodesics on the polar developable: For the tangent line is $(0, \cos\ \gamma,\ \sin\ \gamma)$, where $\gamma + T =$constant. The consecutive tangent is

$$(-\cos \gamma\ d\varepsilon, \cos \gamma - \sin \gamma\ d\gamma - \sin \gamma\ dT,\ \sin \gamma + \cos \gamma\ d\gamma + \cos \gamma\ dT)$$

which, since $d\gamma + dT = o$, reduces to

$$(-\cos \gamma d\varepsilon, \cos \gamma, \sin \gamma).$$

Therefore, the direction cosines of the binormal are $(o, \sin \gamma, -\cos \gamma)$. Now, the normal plane to the curve is a tangent plane to the polar developable, therefore the normal to the polar developable at any point is parallel to $(1, 0, 0,)$ and this is perpendicular to $(0, \sin \gamma, -\cos \gamma)$, whence the osculating plane to a point on the evolute is a geodesic on the polar developable.

Lastly, the radii of curvature and torsion of a point on the evolute may be obtained, and we find, since $\Delta s = d\xi$

$$\frac{1}{\rho'} = \frac{\cos\ \gamma}{\rho}\frac{ds}{d\xi},\ \frac{1}{\sigma'} = \frac{\sin\ \gamma}{\rho}\frac{ds}{d\xi},\ \text{where } \xi = \rho \sec \gamma.$$

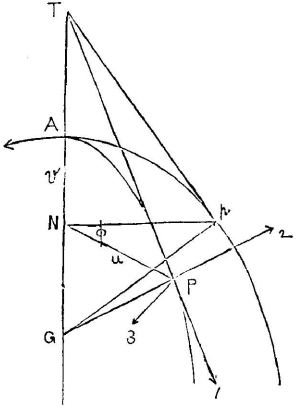
Hence, $\rho'/\sigma' = \tan \gamma$ ; thus if a curve is a geodesic on a cylinder its involutes are plane curves.

§ 18. *Curves on a surface of revolution.*

Let the meridian curve be such that the coordinates of any point $p$ are $(v, u)$, where $AN = v$, $pN = u$. Let $p\,T$ the tangent make an angle $\psi$ with the axis of rotation, so that $\sin \psi = du/ds$, $\cos \psi = dv/ds$.

We see that $d\psi$ is negative for the figure, as drawn; thus

$$\frac{1}{\rho}=-\frac{d\psi}{ds}=\frac{d^2v}{ds^2}\Big/\frac{du}{ds}=-\frac{d^2u}{ds^2}\Big/\frac{dv}{ds}.$$




If P is the position of $p$ after a rotation through of an angle $\phi$, take moving axes (1) along the tangent TP, (2) along the normal GP, (3) along the perpendicular to the meridian plane in the direction of increasing $\phi$, so that we have a right-handed system.

It is obvious that $\xi=ds,\ \eta=0,\ \zeta=ud\phi$.

For the infinitesimal rotations $\partial_1\,\partial_2\,\partial_3$ proceed as follows: the direction cosines of the axis of rotation are $(\cos\psi,\ -\sin\psi,\ 0)$, and this is fixed in space. Therefore

$$d\cos\psi+\sin\psi\,\partial_3=0$$
$$-d\sin\psi+\cos\psi\,\partial_3=0$$
$$-\cos\psi\,\partial_2-\sin\psi\,\partial_1=0$$

whence $$\partial_3=d\psi=-ds/\rho$$

Again, the point A $(-u\sin\psi-v\cos\psi,\ -u\cos\psi+v\sin\psi,\ 0)$ is fixed in space. Therefore

$$\xi-d(u\sin\psi+v\cos\psi)-(-u\cos\psi+v\sin\psi)\partial_3=0$$
$$\eta+d(-u\cos\psi+v\sin\psi)-(u\sin\psi+v\cos\psi)\partial_3=0$$
$$\zeta+(u\sin\psi+v\cos\psi)\partial_2+(-u\cos\psi+v\sin\psi)\partial_1=0.$$

The first two equations simply reduce to $\xi=ds,\ \eta=0$; and the third, combined with $\cos\psi\partial_2+\sin\psi\partial_1=0$, gives

$$\partial_1=\cos\psi\,d\phi,\ \partial_2=-\sin\psi\,d\phi.$$

We have, therefore, the system

$$\xi = ds,\ \eta = o,\ \zeta = ud\phi,\ \partial_1 = \frac{dv}{ds}.\ d\phi,\ \partial_2 = -\frac{du}{ds}.\ d\phi,\ \partial_3 = -\frac{1}{\rho}ds.$$

*Notation* : The element of arc of the curve we are considering will be denoted by $\Delta s$, the angle between two consecutive tangents by $\Delta \varepsilon$, the angle between two consecutive osculating planes by $\Delta T$, the radii of curvature and torsion by $\rho'$, $\sigma'$.

§ 19. *The radii of curvature and torsion of curves traced on a sphere.*

Let $\theta$ be the angular distance of P measured from the pole, so that $\theta + \psi = \pi/2$. Then $u = a \sin\theta$, $v = a\,(1-\cos\theta)$, $\partial_1 = \sin\theta . d\phi$, $\partial_2 = -\cos\theta d\phi$, $\partial_3 = -d\theta$.

Let the curve cut the meridian at an angle $\chi$. The tangent is $(\cos \chi,\ o,\ \sin \chi)$, and $\Delta s \cos \chi = a d\theta$, $\Delta s \sin \chi = a \sin \theta d\phi$.

The angle of contingence is given by

$$\Delta\varepsilon^2 = (-\sin\chi d\chi - \sin\chi \cos\theta d\phi)^2 + (-\sin\chi \sin\theta d\phi - \cos\chi d\theta)^2$$
$$+ (\cos\chi\ d\chi + \cos\chi\cos\theta\ d\phi)^2$$
$$= (d\chi + \cos\theta\ d\phi)^2 + \frac{1}{a^2}\Delta s^2,$$

whence,

$$\frac{1}{\rho'^2} = \left(\frac{d\chi}{\Delta s} + \cos\theta\frac{d\phi}{\Delta s}\right)^2 + \frac{1}{a^2}$$

The radius of torsion will be most easily determined by use of

$$a^2 = \rho'^2 + \sigma'^2\left(\frac{d\rho'}{\Delta s}\right)^2,$$

since $a$ is the radius of spherical curvature of the curve.

§ 20. *The rhomb line on the sphere which crosses the meridian at an angle* $\alpha$.

Here $\gamma = \alpha$, a constant ; and the equation of the rhomb line is $\tan\alpha d\theta = \sin \theta\ d\phi$.

We have

$$\Delta\varepsilon^2 = \cos^2 \theta\ d\phi^2 + \Delta s^2/a^2$$

whence $\quad a^2/\rho'^{\,2} = \cos^2 \theta \sin^2 \alpha/\sin^2 \theta + 1.$

$$\therefore\quad \rho'^{\,2} = a^2 \sin^2 \theta/(1 - \cos^2\theta \cos^2 \alpha)$$

and $\quad \sigma'^{\,2} = (a^2 - \rho'^{\,2})\left(\frac{\Delta s}{d\theta}\right)^2 \Big/ \left(\frac{d\rho'}{d\theta}\right)^2,$

which reduces without much difficulty to

$$\sigma' = a\,(1 - \cos^2\theta \cos^2 \alpha)/(\sin \alpha \cos \alpha),$$

results given in Wolstenholme, Prob. 2189.

§ 21. *A curve on a sphere makes a constant angle with the axis, find the radii of curvature and torsion.*

The direction cosines of the axis AG referred to the moving axes at P are ($\sin\theta, -\cos\theta, o$). The tangent is ($\cos\chi, o, \sin\chi$). Whence

$$\sin\theta\cos\chi=\cos\alpha, \text{ where } \alpha \text{ is the constant angle.}$$

Thus

$$\sin\chi\, d\chi=\cos\theta\,\mathrm{cosec}^2\theta\cos\alpha\, d\theta,\quad a d\theta=\Delta s\cos\chi,$$
$$a\sin\theta\, d\phi=\Delta s\sin\chi.$$

We have

$$\Delta\varepsilon^2=(d\chi+\cos\theta\, d\phi)^2+\Delta s/a^2$$

whence

$$\frac{a^2}{\rho'^2}=\left\{\frac{\cos\theta\cos\alpha}{\sin^2\theta\sin\chi}\cos\chi+\frac{\cos\theta}{\sin\theta}\ n\chi\right\}^2+1$$

Substituting for $\cos\alpha$ its value $\sin\theta\cos\chi$, this becomes

$$\frac{a^2}{\rho'^2}=1+\frac{\cos^2\theta}{\sin^2\theta}\frac{1}{\sin^2\chi}=1+\frac{\cos^2\theta}{\sin^2\theta-\cos^2\alpha}.$$

Whence $\rho'=a\,\mathrm{cosec}\,\alpha\sqrt{\sin^2\theta-\cos^2\alpha}$, and $\sigma'=\rho'\tan\alpha$. These results were given in a Trinity College Examination in 1891.

§ 22. *The curves on a sphere which cut the meridians at angles, $\theta$, $\phi$, respectively.*

For curves cutting the meridian at an angle $\theta$, we have

$$a\, d\theta=\Delta s\cos\theta,\ a\sin\theta\, d\phi=\Delta s\sin\theta,$$

whence

$$d\theta=\cos\theta\, d\phi,\ \Delta s=a d\theta,\ \chi=\theta.$$

Thus

$$\frac{a^2}{\rho'^2}=1+4\cos^2\theta, \text{ and } \sigma'=\frac{a(1+4\cos^2\theta)}{2\sin\theta\cos\theta}.$$

For curves cutting the meridian at an angle $\phi$, we have

$$a\, d\theta=\Delta s\cos\phi,\ a\sin\theta d\phi=\Delta s\sin\phi,\ \chi=\phi\,;$$

whence

$$\sin\phi\, d\theta=\cos\phi\sin\theta\, d\phi.$$

Integrating

$$\sin\phi=k\tan\frac{\theta}{2},$$

where $k$ is constant.

Thus

$$\frac{a^2}{\rho'^2}=1+(1+\cos\theta)^2\left(\frac{ad\phi}{\Delta s}\right)^2$$
$$=1+\frac{1+\cos\theta}{1-\cos\theta}\sin^2\phi.=1+k^2$$
$$\therefore \rho'=a\,(1+k^2)^{-\frac{1}{2}}, \sigma'=\infty,$$

the curves being circles of radii $a\,(1+k^2)^{-\frac{1}{2}}$, a result easily established by stereographic projection.

§ 23. *To find the radius of geodesic curvature of a curve on a sphere.*

The angle of geodesic contingence is the angle between two consecutive geodesic tangents to the curve; it is therefore the angle between two consecutive great circle tangents to the curve, or is the angle between the straight lines perpendicular to these tangent great circles.

The tangent is $(\cos\chi, o, \sin\chi)$ ; the perpendicular to this in the plane is $(\sin\chi, o, -\cos\chi)$.

$$\therefore\ \Delta\varepsilon_g^2=(d\chi+\cos\theta\, d\phi)^2+(\cos\chi\sin\theta\, d\phi-\sin\chi\, d\theta)^2,$$

whence $$\frac{1}{\rho_g'^2}=\left(\frac{d\chi}{\Delta s}+\cos\theta\frac{d\phi}{\Delta s}\right)^2=\frac{1}{\rho'^2}-\frac{1}{a^2}$$

where $\rho_g'$ is the radius of geodesic curvature.

We may proceed to show, as in Picard, *Traite D' Analyse*, Tome I, p. 379, that the only curves on a sphere for which the radius of geodesic curvature is constant are circles.

For, if $\rho_g$ is constant, so must $\rho'$ be constant; and if $\sigma'$ is not infinite, we must have $\rho'=a$, since $a^2=\rho'^2+\sigma'^2\ (d\rho'/\Delta s)^2$. But $\rho'=a$ gives $\rho'_g=\infty$ or $d\varepsilon_g=o$, corresponding to a great circle. For $\sigma'=\infty$, the curves are plane and therefore small circles.

§ 24.. *Geodesics on a surface of revolution.*

Let $\chi$ be the angle the curve makes with the meridian. Denote by $\nu$ the length of the normal PG intercepted between the point and the axis of revolution.

We have for the moving axes : $\xi=ds$, $\eta=o$, $\zeta=ud\phi$, $\partial_1=v'd\phi$, $\partial_2=-u'd\phi$, $\partial_3=-ds/\rho$, where $u', v'$ stand for $du/ds$, $dv/ds$ respectively.

Also $\nu=u\sec\phi$, or $1/\nu=v'/u$. The consecutive tangent to $\cos\chi$, $(o, \sin\chi)$ is

$$\cos\chi-\sin\chi\, d\chi+\sin\chi\,\partial_2,\ -\sin\chi\,\partial_1+\cos\chi\,\partial_3,\ \sin\chi+\cos ds\,\chi\ \chi-\cos\chi\,\partial_2$$

For a geodesic, the binormal is perpendicular to the tangents and to the principal normal (o, 1, o).

Thus, the binormal is $(\sin\chi, o, -\cos\chi)$, and if this is perpendicular to the consecutive tangent, we have

$$-d\chi+\partial_2=o.$$

$\therefore$ $$d\chi=-d\phi\ .\ du/ds$$

Also $$\Delta s\cos\chi=ds,\ \Delta s\sin\chi=ud\phi$$

$$\therefore\ \tan\chi=ud\phi/ds, \text{ or } d\chi/\tan\chi+du/u=o.$$

Integrating, $u \sin\chi = \text{constant}$: the characteristic property of a geodesic on a surface of revolution.

For the radii of curvature and torsion of a geodesic on a surface of revolution, we have

$$\Delta\varepsilon^2 = (-\sin\chi\partial_1 + \cos\chi\partial_3)^2.$$

$$\therefore \quad 1/\rho' = \sin^2\chi v'/u + \cos^2\chi/\rho = \sin^2\chi/\nu + \cos^2\chi/\rho$$

The binormal is $(\sin\chi, o, -\cos\chi)$, whence with the use of the condition $d\chi - \partial_2 = o$, we obtain

$$\Delta T^2 = (\cos\chi\partial_1 + \sin\chi\partial_3)^2,$$

whence $\quad 1/\sigma' = \sin\chi\cos\chi(1/\nu - 1/\rho)$

The former result is identical with Euler's Theorem, and the latter with Frost, § 751.

§ 25. The method *apparently* fails for the hyperboloid of revolution. For, the geodesics are generators, for which $\rho' = \infty$, $\sigma' = \infty$. The generators are, in fact, given by

$$d\chi = \partial_2, \; -\sin\chi\partial_1 + \cos\chi\partial_3 = o,$$

when the consecutive tangent is identical in direction with the first tangent, and the binormal does not exist.

We get in this case

$$u \sin\chi = k, \; \tan\chi = u d\phi/ds = -(1/\rho) ds/v' d\phi$$

$$\therefore \quad \frac{k^2}{u^2 - k^2} = -\frac{1}{\rho}\frac{u}{v'}, \text{ or } \rho = -\frac{u(u^2 - k^2)}{v' k^2}$$

To integrate this meridian curve, writing $x$, $y$ for, $u$ $v$, we have

$$\frac{\dfrac{d^2y}{dx^2}}{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}} = -\frac{k^2}{x(x^2 - k^2)}.$$

$$\therefore \quad \frac{d^2y}{dx^2}\left\{\frac{1}{\dfrac{dy}{dx}} - \frac{\dfrac{dy}{dx}}{1 + \left(\dfrac{dy}{dx}\right)^2}\right\} = \frac{1}{x} - \frac{1}{2(x-k)} - \frac{1}{2(x+k)}$$

Integrating

$$\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} = A\,x(x^2 - k^2)^{-\frac{1}{2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{A\,x}{\{(1 - A^2)x^2 - k^2\}^{\frac{1}{2}}}$$

$$\therefore \quad y = \frac{A}{(1 - A^2)}\sqrt{(1 - A^2)x^2 - k^2} + B$$

If we write $u=a\cosh t$, $v=\text{constant}-b\sinh t$, it may easily be verified that along a generator

$$\cosh t.\sin\chi=a/\sqrt{(a^2+b^2)}$$

In fact

$$\rho=(a^2\sinh^2 t+b^2\cosh^2 t)^{\frac{1}{2}}/ab$$

and the expression $-u(u^2-k^2)/v'k^2$ reduces to this when $k^2=a^4/(a^2+b^2)$.

This theorem is, of course, an investigation of the obvious theorem that the only surface of revolution, on which a rectilinear generator exists, is the hyperboloid of revolution including the cylinder and cone.

§ 26. For the radius of geodesic curvature of a curve on a surface of revolution, we may use the theorem that the angle of geodesic contingence is equal to the angle between the tangent at any point and the projection of the consecutive tangent on the tangent plane to the surface at the former point.

The tangent being

$$x/\cos\chi=y/o=z/\sin\chi,$$

the consecutive tangent is

$$\frac{x-\Delta s\cos\chi}{\cos\chi-\sin\chi\,d\chi+\sin\chi\partial_2}=\frac{y}{-\sin\chi\,\partial_1+\cos\chi\partial_3}$$
$$=\frac{z-\Delta s\sin\chi}{\sin\chi+\cos\chi d\chi-\cos\chi\,\partial_2}$$

The projection of this on the plane $y=o$, is

$$\frac{x-\Delta s\cos\chi}{\cos\chi-\sin\chi d\chi+\sin\chi\partial_2}=\frac{z-\Delta s\sin\chi}{\sin\chi+\cos\chi d\chi-\cos\chi\partial_2}$$

$$\therefore\ \Delta\bar{\epsilon}=d\chi-\partial_2=d\chi-u'd\phi=d\chi-\tan\chi\,du/u$$
$$=d(u\sin\chi)/u\cos\chi,$$ as in Frost, p. 329, Ex. (3).

The radius of geodesic curvature is

$$1/\rho_g=\Delta\bar{\epsilon}/\Delta s=d\,(u\sin\chi)/uds.$$

This expression becomes indeterminate for the parallels of the surface of revolution, for which $\chi=\pi/2$. Take however $\Delta\bar{\epsilon}=d\chi-u'd\phi=-u'd\phi, \Delta s=ud\phi$; and we obtain

$$\frac{1}{\rho_g}=-\frac{u'}{u},$$

whence $\rho_g=-u\operatorname{cosec}\phi=-\text{PT}$.

§ 27. *The only surface of revolution on which the loxodromes have a constant geodesic curvature is the pseudo-sphere.*

If $\chi$ is constant, we have $du/uds$=constant. The meridian curve is therefore given by

$$u dy/y = \sqrt{dx^2+dy^2}$$

$$\therefore \qquad x = \int \frac{\sqrt{a^2-y^2}\cdot dy}{y}$$

or $y = a \sin \theta$, $x = a\,(\log \tan\frac{\theta}{2} + \cos \theta)$, which are the equations of the tractrix.

§ 28. *The radius of curvature of any curve on a surface of revolution.*

From the tangent and consecutive tangent written down in § 26, we obtain

$$\triangle\, \varepsilon^2 = (d\,\chi - \partial_2)^2 + (\sin \chi\, \partial_1 - \cos \chi\, \partial_3)^2$$

whence

$$\frac{1}{\rho'^2} = \left(\frac{d\chi}{\triangle s} + \frac{\tan \chi}{u} \cdot \frac{du}{\triangle s}\right)^2 + \left(\frac{\sin^2 \chi}{\nu} + \frac{\cos^2 \chi}{\rho}\right)^2$$

§ 29. The asymptotic lines (or *chief lines* according to Frost) for a surface of revolution are given by the condition that the principal normal of the surface (0, 1, 0) is perpendicular to the consecutive tangent to the curve. Hence

$$- \qquad \sin \chi\, \partial_1 + \cos \chi\, \partial_3 = 0.$$

$$\therefore \qquad \sin^2 \chi/\nu + \cos^2 \chi/\rho = 0.$$

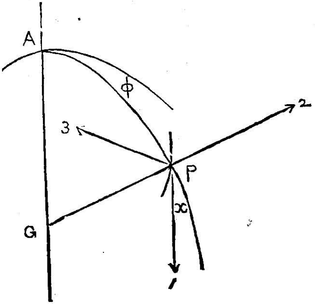
The surface of revolution for which the asymptotic lines are at right angles at every point, is the catenoid, or surface of revolution generated by the revolution of the catenary about its directix.

For, $\tan \chi = \pm\sqrt{-\frac{\nu}{\rho}}$, and if these two directions are perpendicular, we have $\nu + \rho = 0$, the characteristic property of the catenary. This example occurs in Frost, p. 374, Ex. 5.

§ 30. I propose to consider a second set of moving axes on the surface of revolution formed by a straight line in the tangent plane making an angle $\chi$ with the meridian, the principal normal to the surface and the perpendicular to these two.

The coordinates of P in the meridian curve are $(v, u)$, $ds$ is the element of arc of the meridian curve at P, $\triangle s$ the element of arc of the curve on the surface which has the axis of $x$ as a tangent. Thus $ds = \triangle s \cos \chi$, $u d\phi = \triangle s \sin \chi$, $\xi = \triangle s$, $\eta = 0$, $\zeta = 0$.

If $(x, y, z)$ are the coordinates of any point and (X, Y, Z) the coordinates of the same point referred to the axes at P of § 18, we have




$$x = X \cos \chi + Z \sin \chi$$
$$y = Y$$
$$z = Z \cos \chi - X \sin \chi.$$

The direction cosines of the axis of revolution were, referred to the axes of § 18, $(\cos \psi, -\sin \psi, o)$ and by considering a point at an infinite distance in this direction, the direction cosines of the axis of revolution referred to our new axes are

$$(\cos \psi \cos \chi, -\sin \psi, -\cos \psi \sin \chi).$$

This being fixed in space, we have

$$d\,(\cos \psi \cos \chi) + \sin \psi\, \partial_3 - \cos \psi \sin \chi\, \partial_2 = o$$
$$-d \sin \psi + \cos \psi \sin \chi\, \partial_1 + \cos \psi \cos \chi\, \partial_3 = o$$
$$-d\,(\cos \psi \sin \chi) - \cos \psi \cos \chi\, \partial_2 - \sin \psi\, \partial_1 = o.$$

Only two of these equations are independent as is seen by multiplying them by $\cos \psi \cos \chi, -\sin \psi, -\cos \psi \sin \chi$ and adding. For a further condition consider the point G whose coordinates are $(o, -u \sec \psi, o)$. it is displaced along the axis of revolution a distance

$$d\,(v + u \tan \psi) = \left(\frac{1}{\cos \psi} - \frac{u}{\rho \cos^2 \psi}\right) ds.$$

The coordinates of the displaced position are therefore,

$$\cos \chi \left(1 - \frac{u}{\rho \cos \psi}\right) ds, -u \sec \psi - \tan \psi \left(1 - \frac{u}{\rho \cos \psi}\right) ds,$$
$$-\sin \chi \left(1 - \frac{u}{\rho \cos \psi}\right) ds.$$

We therefore obtain the conditions

$$\cos \chi \left(1 - \frac{u}{\rho \cos \psi}\right) ds = \Delta s + u \sec \psi\, \partial_3,$$
$$-\tan \psi \left(1 - \frac{u}{\rho \cos \psi}\right) ds = -d\,(u \sec \psi),$$

$$-\sin\chi\left(1-\frac{u}{\rho\cos\psi}\right)ds=-u\sec\psi\,\partial_1,$$

with
$$ds=\Delta s\cos\chi.$$

Therefore
$$\partial_1=\sin\chi\cos\chi\left(\frac{1}{\nu}-\frac{1}{\rho}\right)\Delta s,$$

$$\partial_3=-\left(\frac{\sin^2\chi}{\nu}+\frac{\cos^2\chi}{\rho}\right)\Delta s,$$

where $\nu=u\sec\psi=$ the normal PG.

For $\partial_2$ we can now use the equation
$$d(\cos\psi\cos\chi)+\sin\psi\,\partial_3-\cos\psi\sin\chi\,\partial_2=0,$$
giving
$$-\cos\psi\sin\chi d\chi+\sin\psi\cos\chi\frac{\Delta s\cos\chi}{\rho}$$
$$-\sin\psi\left(\frac{\sin^2\chi}{\nu}+\frac{\cos^2\chi}{\rho}\right)\Delta s-\cos\psi\sin\chi\,\partial_2=0,$$

whence
$$\partial_2=-\left(\frac{d\chi}{\Delta s}+\tan\psi\frac{\sin\chi}{\nu}\right)\Delta s.$$

Thus our set of equations for the moving axes now adopted are:

$$ds^2=du^2+dv^2,\ \tan\psi=\frac{du}{dv},\ \rho=-\frac{ds}{d\psi},\ \nu=u\sec\psi,$$
$$\xi=\Delta s,\ \eta=0,\ \zeta=0,\ ds=\Delta s\cos\chi,\ ud\phi=\Delta s\sin\chi,$$
$$\partial_1=\sin\chi\cos\chi\left(\frac{1}{\nu}-\frac{1}{\rho}\right)\Delta s,\ \partial_2=-\left(\frac{d\chi}{\Delta s}+\tan\psi\frac{\sin\chi}{\nu}\right)\Delta s,$$
$$\partial_3=-\left(\frac{\sin^2\chi}{\nu}+\frac{\cos^1\chi}{\rho}\right)\Delta s.$$

§ 31. *The radii of curvature and torsion of any curve on a surface of revolution.*

Let the curve make an angle $\chi$ with the meridian, and take moving axes as in § 30.

The consecutive tangent to (1, 0, 0) is $(1, \partial_3, -\partial_2)$, whence
$$\Delta\varepsilon^2=\partial_3^2+\partial_2^2$$
and
$$\frac{1}{\rho'^2}=\left(\frac{\sin^2\chi}{\nu}+\frac{\cos^2\chi}{\rho}\right)^2+\left(\frac{d\chi}{\Delta s}+\tan\psi\frac{\sin\chi}{\nu}\right)^2,$$
agreeing with the result of § 28.

The binormal is perpendicular to (1, 0, 0) and $(1, \partial_3, -\partial_2)$.

Its direction-cosines are proportional to $(0, \partial_2, \partial_3)$ and are therefore $(0, \cos\lambda, \sin\lambda)$, where
$$\tan\lambda=\left(\frac{\sin^2\chi}{\nu}+\frac{\cos^2\chi}{\rho}\right)\Big/\left(\frac{d\chi}{\Delta s}+\tan\psi\frac{\sin\chi}{\nu}\right).$$

Thus $\Delta T^2=(-\cos\lambda.\,\partial_3+\sin\lambda.\,\partial_2)^2+(-\sin\lambda\, d\lambda-\sin\lambda\,\partial_4)^2$
$+(\cos\lambda\, d\lambda+\cos\lambda\,\partial_1)^2$

or
$$\Delta T=d\lambda+\partial_1$$

and
$$\frac{1}{\sigma'}=\frac{d\lambda}{\Delta s}+\sin\chi\cos\chi\left(\frac{1}{\nu}-\frac{1}{\rho}\right).$$

In a geodesic $\lambda=\frac{\pi}{2}$, and the radius of torsion of a geodesic is $\sin\chi\cos\chi\left(\frac{1}{\nu}-\frac{1}{\rho}\right)$ as in § 24.

§ 32. *The loxodromes on a surface of revolution.*

Here $\chi=\text{constant}=\alpha$.

The radius of curvature is given by

$$\frac{1}{\rho'^2}=\left(\frac{\sin^2\alpha}{\nu}+\frac{\cos^2\alpha}{\rho}\right)^2+\frac{\sin^2\alpha}{\nu^2}\left(\frac{\nu^2}{u^2}-1\right),$$
$$=\frac{\sin^2\alpha}{u^2}+\frac{2\sin^2\alpha\cos^2\alpha}{\nu\rho}+\frac{\cos^4\alpha}{\rho^2}-\frac{\sin^2\alpha\cos^2\alpha}{\nu^2}.$$

The radius of torsion is given by

$$\frac{1}{\sigma'}=\frac{d\lambda}{\Delta s}+\sin\alpha\cos\alpha\left(\frac{1}{\nu}-\frac{1}{\rho}\right),$$

where
$$\tan\lambda=\left(\frac{\sin^2\alpha}{\nu}+\frac{\cos^2\alpha}{\rho}\right)\Big/\frac{\tan\psi\sin\alpha}{\nu}.$$

These results are given in Resal, p. 145, with an error in the result for the radius of curvature.

Considering a loxodrome on a circular cone of semi-vertical angle $\beta$ these expressions give, since $1/\rho=0$, $\nu=u\sec\beta$, $\psi=\beta$,

$$\rho'=u\operatorname{cosec}\alpha\,(1-\cos^2\alpha\cos^2\beta)^{-\frac{1}{2}}$$

and
$$\tan\lambda=\sin\alpha\cot\beta,$$

whence the osculating plane makes a constant angle with the tangent plane to the cone, and $1/\sigma'=\sin\alpha\cos\alpha\cos\beta/u$.

If in these expressions we put $\beta=0$, they reduce to the well known formulae for the circular helix.

§ 33. The asymptotic lines are given by the condition that the osculating plane coincides with the tangent plane to the surface, whence $\lambda=0$.

Thus for the asymptotic lines

$$\frac{\sin^2\chi}{\nu}+\frac{\cos^2\chi}{\rho}=0$$
$$1/\sigma'=\sin\chi\cos\chi\left(\frac{1}{\nu}-\frac{1}{\rho}\right),$$

whence we obtain

$$\sigma'=\sqrt{-\nu\rho}$$

a special case of a more general theorem due to Euneper.

# REVIEWS:

*Systemes Cinematiques*, (Scientia Series No. 31): by Dr. L. Creliers (Gauthier Villars, Paris, 1911), Price, 2 francs.

Besides the purely geometrical and the purely analytical methods of tracing curves there exists a third one, *viz.* the kinematical method. Kinematical Geometry differs from Kinematics only in this that in it time, hence also velocity and acceleration, are not considered. In kinematical geometry the relative motion of two planes, superposed one upon the other, is given. If an element fixed in one of the two planes is given, we can find from the relative motion of the two planes the curve which the element determines in the other plane. The motion of a plane can be determined either by the paths of two points in it, or by the path of a fixed point and the envelope of a fixed line, or by the envelopes of two fixed curves; finally also by the rolling, without sliding, of a curve fixed in the moving plane on a curve in the resting plane. For a motion of the last mentioned kind the following theorems can be proved :—

(i) At any moment of the motion the normals to the paths (trajecttoire) of all the points pass through the same point, the instantaneous centre J.

(ii) The locus of the instantaneous centres (centrodes) in the resting plane is a curve (base, space-centrode), and in the moving plane also a curve (roulante, body-centrode).

(iii) At any moment the point of contact of these two curves is the instantaneous centre J.

(iv) If from J we draw the normals to any curve C fixed in the moving plane, then these normals meet the curve C at the points of contact of C with its envelope.

It is also clear that in this case it does not matter which of the two planes we consider as resting and which as moving.

The different conditions determining the motion of a plane can be realised by properly devised mechanisms (systémes articulés). In the little book under review the aim of the author is the study, from a geometrical standpoint, of the more simple arrangements which form the basis of these mechanisms. From a geometrical point of view the book is highly interesting and contains a great deal of information. As far as possible, the different arrangements are all dealt with in a uniform way and with great thoroughness.

A summary of the main results arrived at in the book will perhaps interest some of the readers :

Ch. I. A right angle OLP moves so that one side OL always passes through a fixed point O whilst the vertex L of the angle slides along a fixed straight line O′L. The distance OO′ of the fixed point from the fixed line is $a$.

1. The space-centrode (base) is a parabola :—

$$y^2 = -ax$$

The body-centrode (roulante) is the curve :

$$x^4 - a^2(x^2+y^2) = o.$$

This curve has two branches, of which the one to the left rolls on the parabola. The origin of co-ordinates is taken at O and the $y$—axis is parallel to the fixed line.

2. The path (trajectoire) of a point :—

Every point on OL describes a conchoid, every point on LP describes an orthoconchoid. Any point Q in the moving plane may be taken as lying on the side LQ of an angle $\alpha =$ QLP. The path of Q is an oblique conchoid whose equation is :—

$$[xy \cos\alpha + (x^2+ax-l^2) \sin\alpha]^2 = (x\cos\alpha - y\sin\alpha + a \cos\alpha)^2 (l^2-a^2),$$

where $l =$ QL. The origin is taken at O′. For properly chosen values of $\alpha$ and $l$, we can easily get from this equation the path of any point in the moving plane.

3. The envelope of the moveable line LP is a parabola :—

$$y^2 = -4\,ax,$$

the origin of coordinates being taken at O.′

The envelope of a line parallel to LP is a curve parallel to the parabola.

The envelope of a line parallel to OL is a circle with its centre at O

The envelope of any line LQ has the equation :—

$$x^2+y^2 = (2\,a \sin\alpha - x \sin\alpha + y \cos\alpha)^2,$$

where $\alpha =$ OLQ and the origin of coordinates is taken at O.

Finally, the envelope of any line parallel to LQ is a curve parallel to the above parabola.

4. It has been mentioned above that any point on OL describes a conchoid. Take OL in a given position and consider the tangents to the conchoids at all the points of intersection of OL with the conchoids The envelope of all these tangents is a parabola :—

$$y^2 = -4\,mx$$

where $m =$ OJ, J being the origin of coordinates, and OJ the axis of $x$.

For every other position of OL we get a similar equation. The locus of the foci of all these parabolas is another parabola. Also any point on

LP describes an orthoconchoid, and the envelope of LP is a parabola. If we take a fixed position of LP the tangents to all the orthoconchoids at the successive points along LP envelope a parabola :—

$$y^2=-4\,nx,$$

where $n=$JP, JP being the axis of $x$.

The locus of the foci of the different parabolas corresponding to different positions of LP is also a parabola.

5. The evolute of the parabola $y^2=-4ax$ is

$$y^2=-\frac{8}{27}\frac{1}{2a}(x+2a)^3,$$

and its pedal curve with respect to O′ as origin is a cissoid

$$\rho=a\tan\phi\sin\phi.$$

Ch. II. A right angle OLP moves in such a way that the side OL always passes through the fixed point O whilst a given point P on the side LP slides along a fixed straight line passing through O.

Let $a$ be the distance of the point P from the vertex L.

1. The equation of the space centrode is

$$x^4=a^2\,(x^2+y^2),\text{ origin at O},$$

an equation similar to that of the space-centrode in the preceding arrangement.

The equation of the body-centrode is :—

$$y^2=ax,$$

the origin being taken at O′ on the fixed line. OO′$=a$.

2. The envelope of LP is, with the origin at O :—

$$x^6+x^4y^2-a^2\,(16\,y^4+3\,x^4+20\,x^2\,y^2)-a^4\,(8\,y^2-3\,x^2)-a^6=0.$$

Or, in tangential coordinates

$$v^2=\frac{a^2u^2}{1-a^2u^2}$$

The equations of the other envelopes become even more complicated.

3. The path of the vertex L of the right angle is given by :—

$$a^2x^2=y^2(x^2+y^2).$$

Any point on QL describes the curve

$$(ax+by)^2=y^2(x^2+y^2),$$

$b$ being the distance of the point from L.

Any point Q on LP traces out a curve

$$(y^2-k^2-ak)^2=x^2(k^2-y^2),$$

where $k=$PQ.

CH. III. A right angle OLP or OLD moves so that the side OL always passes through the fixed point O, and a fixed point P on the side LP slides along a fixed straight line at a given distance from O.

Let $OO' = 2R$ be the constant distance of the fixed line from the fixed point O. The direction of the fixed line is the direction of the $y$ axis. Let $O''$ be the mid-point of $OO'$. LP and LD are of constant length.

1. The space-centrode has the equation

$$y^2 = 4Rx, \text{ (origin at } O'').$$

The body-centrode is given by

$$y^2 = -4Rx, \text{ (origin at } O'').$$

2. The vertex of the angle OLP traces out a strophoid whose equation is

$$x(x^2+y^2)+2R(y^2-x^2)=0, \text{ (origin at } O').$$

Any point Q of the moving plane describes a curve whose equation is

$$[xy \sin\phi - (x^2 - 2Rx - l^2)\cos\phi + 2Rl]^2 = (y\cos\phi + x\sin\phi - 2R\sin\phi)^2(l^2 - x^2),$$

where $l = DQ$, $\phi = \pi - LDQ$, and $O'$ is the origin.

3. The envelope of LP is

$$y^2(y^2+x^2) - 2Rx(8x^2+9y^2) - 27R^2y^2 = 0,$$

$O''$ being the origin.

Some more envelopes and also the motion of a right angle with the vertex on a parabola and one side passing through the focus of the parabola are discussed in this chapter.

CH. IV. The right angle OLP moves so that OL always goes through the fixed point O and the vertex L describes a circle round another fixed point $O_1$.

Let $O_1L = R$, $OO_1 = d$.

1. The space-centrode has the equation

$$R^2[y^2+x(x-d)]^2 - d^2(x-d)^2(x^2+y^2) = 0, \text{ (origin at } O_1)$$

The body-centrode is given by

$$4R^2x^4 = (x^2+y^2)(R^2+x^2-d^2)^2,$$

the origin being taken at A, the point of intersection of the given circle with the line $OO_1$, $(O_1\ O\ A)$.

2. The envelope of LP is given by the equation

$$x^2(R^2-d^2)+y^2R^2 = (R^2-d^2)R^2 \text{ (origin at } O_1).$$

The envelope of any other straight line of the moving plane is a curve parallel to this conic.

3. Any point Q in the moving plane describes the curve

$$\{[(x-d)\sin\gamma'+y\cos\gamma'](x^2+y^2+l^2-R^2)-2l^2x\sin\gamma'\}^2$$
$$+\{[(x-d)\cos\gamma'-y\sin\gamma'](x^2+y^2+l^2-R^2)+2l^2y\sin\gamma'\}^2$$
$$=4l^2\cos^2\gamma'[(x(x-d)+y^2-dy\tan\gamma']^2,$$

where $\gamma'=\frac{\pi}{2}+LPQ$ and $l=PQ$. The origin of coordinates is at $O_1$ and $OO_1$ is the $x$-axis.

We get an interesting special case by taking the fixed point O on the given circle.

CH. V. A line AB of constant length slides along the two sides $x$ and $y$ of a right angle.

1. The space-centrode is a circle of radius $R=AB$, with its centre at the origin *i.e.*, at the point of intersection of the given straight lines.

The body-centrode is a circle of radius $\frac{1}{2}R$ on the inside of the first circle.

2. Any point M on AB describes an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a=MB$ and $b=MA$.

3. The envelope of AB is an astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=R^{\frac{2}{3}}.$$

The evolute of the astroid is another astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=(2R)^{\frac{2}{3}}.$$

Similar results follow if instead of rectangular axes oblique ones are taken. The envelopes of any line in the moving plane are also dealt with in this chapter.

CH. VI. A line AB of constant length $a$ moves so that one end, say A, slides along a fixed line AX, the other end B describes a circle of radius R round a fixed point O.

Let the fixed line AX pass through O, and let O be the origin.

1. The space-centrode has the equation

$$(x^2+y^2)(a^2-R^2-x^2)^2=4R^2x^4.$$

The equation of the body-centrode is

$$R^2(x^2+y^2-ax)^2=a^2x^2[y^2+(x-a)^2].$$

2. Any point P in the moving plane describes the curve

$$[n \cos \gamma (x^2+y^2+m^2-R^2)-2 my^2]^2$$
$$+[n \sin \gamma (x^2+y^2+m^2-R^2)-2 mxy]^2$$
$$=4 m^2n^2 (y \sin \gamma - x \cos \gamma)^2,$$

where $m=AP$, $n=BP$, $a=AB$, $\gamma=APB$.

3. The envelope of AB in tangential coordinates has the equation

$$(u^2+v^2) [1+u^2 (a-R^2)]^2=4 a^2 u^2 v^2.$$

The envelope of any line AP, making an angle $\beta$ with AB is given by

$$(u^2+v^2) [1+u^2 (a^2-R^2)]^2=4 a^2 u^2 (u \sin \beta + v \cos \beta)^2.$$

Any other line in the moving plane can be taken as a line parallel to AP and its envelope found.

It is to be regretted that the author has not given a short introductory chapter on Kinematical Geometry, which doubtless would have made the interesting book more easy reading. We regret also the absence of clear definitions. Almost all the figures are overladen with details and it is a hard task to find out from the amazing number of lines and letters what one is looking for. Some additional figures with fewer details would have been of great help to the reader.

Obviously the book is not meant for a first reading and supposes a certain amount of acquaintance with the elements of the subject.

A. STEICHEN.

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"*Arithmetique Generale.*" par E. Dumont; (Hermann and Fils, Paris, 1911, pp. xvii+275; 10 francs).

The author of this book defines "Arithmetic" as the science of numbers. "Number" is not defined, but is taken to be almost a fundamental concept intimately connected with the measure of geometrical magnitudes in one, two, and three dimensions. The term "arithmetic" as thus defined is, practically, synonymous with "Universal Algebra"; but the book under review confines itself to a study of the simpler operations (addition, subtraction, multiplication, division, involution and evolution) as applied to the following classes of numbers: (1) "natural" or "absolute" numbers, *i.e.*, the scalars of ordinary arithmetic; which may be commensurable (whole or fractional) or incommensurable; (2) "relative" numbers which are divided into four sub-classes: (i) "monaries" or real numbers of ordinary algebra; (ii) "binaries" or the complex numbers of ordinary algebra; (iii) "ternions"; and (iv) "quater-

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nions". "Ternions" are merely quaternions with one non-scalar element zero; *lame* quaternions, so to speak. They are neither useful nor pretty, nor even capable of standing by themselves, as the simplest operation (addition) results, in general, in a quaternion. They seem to have been invented to fill the gap between binaries and quaternions. If it was necessary to fill that gap, the author might have given us a resumé of the various systems of vector analysis which differ from quaternions in rejecting the scalar element of the latter, and which are as incapable of a satisfactory study as algebraic square roots are when the ancient practice of dodging $\sqrt{-1}$ is adopted.

Another defect we have to find in the book is the frequent, but unnecessary change of terminology. For instance, the Hamiltonian $i, j, k$ are replaced by $i, j, r$; and it is not explained what offence poor $k$ was guilty of.

The author does not treat of bi- and tri-quaternions, on the ground that he cannot find what they serve to measure. This is a regrettable omission, as information regarding these subjects is not readily available. As for applications, biquaternions, (by which, we presume, are meant Clifford's biquaternions or Octonions, and not Hamilton's) have been applied to measure localized vectors, screws and wrenches. Triquaternions we know of, as forming a system of Algebra with a special multiplication table, but we have not been able to obtain the memoirs where their applications are explained.

Coming now to the merits, the book is well (even brightly) written and explanations are lucid, in spite of the medieval passion for special nomenclature which leads the author to invent unnecessary names—we had almost said, nicknames. In some cases the method of approach is novel and suggestive. The book having been written for beginners the treatment is not exhaustive. Readers desirous of obtaining a general knowledge of the subject will find the book useful.

BRAMG.

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## SHORT NOTES.

### The solution of the algebraical equation $f(x)=0$ in two particular cases.

1. Suppose $f(x)$ to be of even degree $2n$, and let every one of its $n$ pairs of roots satisfy the relation $\alpha_r+\beta_r=2h$ $[r=1, 2, \ldots n]$.

Then $f(x)\equiv\Pi_n^1\{x^2-(\alpha_r+\beta_r)x+\alpha_r\beta_r\}=\Pi_n^1(x^2-2hx+\alpha_r.\beta_r)$.

If we put $x=2h-y$, we get $\Pi_n^1(y^2-2hy+\alpha_r\beta_r)$ for $f(x)$; so that the expressions coincide, and the roots cannot be obtained by the method for common factors. But $f(x)\equiv\Pi_n^1\{(x-h)^2-(h^2-\alpha_r\beta_r)\}\equiv\Pi_n^1(z^2-h_r)$, suppose. Hence $x=h+z$, and $f(x)$ can be expanded in powers of $z$. Putting $S_1, S_2, S_3 \ldots S_n$ for the sums one, two,... $n$ at a time of the quantities $h_r$, we have the identity

$$z^{2n}-S_1z^{2n-2}+S_2z^{2n-4}\ldots+(-1)^nS_n$$
$$=f(h)+zf^1(h)+\frac{z^2}{\underline{|2}}f^2(h)\ldots+\frac{z^{2n-1}}{\underline{|2n-1}}f^{2n-1}(h)+\frac{z^{2n}}{\underline{|2n}}f^{2n}(h).$$

We therefore deduce at once

(1) $f^1(h)=f^3(h)=\ldots=f^{2n-1}(h)=0$, and

(2) $\dfrac{f^{2n}(h)}{\underline{|2n}}=1$, and generally $S_r=(-1)^r\dfrac{f^{2n-2r}(h)}{\underline{|2n-2r}}$.

Thus the coefficients of the equation in $z$, which is of the $n^{th}$ degree in $z^2$, are all determined, and we can deduce the $n$ values of $h_r$ and hence of $\alpha_r\beta_r$; and as $\alpha_r+\beta_r=2h$, all the $n$ pairs of roots are determined by means of an equation of the $n^{th}$ degree. We also see that equations (1) give the conditions for which this method will apply.

*Example.* $f(x)\equiv x^6-9x^5+19x^4+21x^3-112x^2+120x-40=0$. Here $f^5(x)\equiv720x-1080$ vanishes for $x=\frac{3}{2}$, for which value $f^1(x)$ and $f^3(x)$ also vanish. Hence the equation is of the stated kind, and $z$ has to be found from

$$z^6-\frac{354}{24}z^4+\frac{179}{16}z^2-\frac{121}{64}=0.$$

This gives $z^2=\frac{1}{4}, \frac{1}{4}(29\pm12\sqrt{5})$, so that $\alpha_r\beta_r=2, -5\pm3\sqrt{5}$; and as $\alpha_r+\beta_r=3$, the solution can be completed.

2. Let every one of the $n$ pairs of roots satisfy the relation $\alpha_r\beta_r=k^2$ $[r=1, 2, \ldots n.]$

Then $f(x)\equiv\Pi_n^1\{x^2-(\alpha_r+\beta_r)x+k^2\}$. If we put $x=k^2/y$, we get for $f(x)$, $k^{2n}/x^{2n}.\Pi_n^1\{k^2-(\alpha_r+\beta_r)x+x^2\}$; so that here also the

equation cannot be depressed by the method of common factors. As $k=o$, we see that the equation is reciprocal for the substitution $k^2/x$. Hence, putting $z=x+k^2/x$, we have

$\prod_n^1 (z-k_r)=o$, where $k_r=\alpha_r+\beta_r$; and we get, as in the first case

$$z^n-S_1 z^{n-1}+S_2 z^{n-2}\ldots+(-1)^n S_n=0.$$

The values of $S_1$, $S_2\ldots S_n$ may be obtained by dividing out by $x^n$ in the given equation, and expressing

$$x+k^2/x,\ x^2+k^4/x^2,\ldots x^n+k^{2n}/x^n,$$

in terms of $z$ by the ordinary rules.

Or, we may put $x=k(\cos\theta+\iota\sin\theta)$, so that $k^2/x=k(\cos\theta-\iota\sin\theta)$ and $z=2k\cos\theta$. Thus we shall have $k^n\ 2\cos n\theta+p_1 k^{n-1}\,2\cos(n-1)\theta+p_2 k^{n-2}2\cos(n-2)\theta\ldots+p_n=0$. Expanding the cosines of the multiple angles and putting $z$ for $2k\cos\theta$, the equation to determine the $n$ values of $k_r$ can be obtained.

*Example.* $f(x)=2x^6-7x^5-50x^4+124x^3+300x^2-252x-432=0$. As $f(-6/x)=o$, gives the same equation, we have $\alpha_r\,\beta_r=-6$, for every one of the three pairs of roots. Remove all terms after the middle term, divide by $x^3$, and put $2(-6)^{\frac{1}{2}}\cos\theta$, $2(-6)\cos 2\theta$, $2(-6)^{\frac{3}{2}}\cos 3\theta$ for $x$, $x^2$, $x^3$ respectively. Thus we obtain

$$4(-6)^{\frac{3}{2}}(4\cos^3\theta-3\cos\theta)-14(-6)(2\cos^2\theta-1)$$
$$-100(-6)^{\frac{1}{2}}\cos\theta+124=0,$$

whence, putting $z=2(-6)^{\frac{1}{2}}\cos\theta$, we derive

$$2z^3-7z^2-14z=40=0.$$

This gives $z=2,\ 4,-5/2$, which are the values of $\alpha_r+\beta_r$; and as $\alpha_r\,\beta_r=-6$, the solution can be completed.

K. J. SANJANA.

## Feuerbach's Theorem.

Let D,E,F be the middle points of the sides of the triangle ABC and suppose $a>b>c$.

Consider the radical axis of the nine-points circle and the in-circle and let $D\alpha$, $E\beta$, $F\gamma$, be the perpendiculars on it. Then by the well-known property of the radical axis, since D is on the nine-points circle,

2 IN. $D\alpha$ = square of the tangent from D to the incircle

$$=\left[\tfrac{1}{2}\{(s-c)-(s-b)\}\right]^2$$
$$=\tfrac{1}{4}(b-c)^2$$

$$\therefore\ D\alpha : E\beta : F\gamma :: (b-c)^2 : (a-c)^2 : (a-b)^2 \qquad (1)$$

Next let P be a point on the smaller arc DF of the nine-points circle, such that

$$DP : FP :: b-c : a-b.$$

By Ptolemy's theorem

$$EF.DP+DE.FP=DF.EP$$

$$\therefore aDP+c.FP=b.EP$$

$$\text{Now } \frac{DP}{b-c}=\frac{FP}{a-b}=\frac{a.DP+c.FP}{a(b-c)+c(a-b)}=\frac{b\,EP}{b(a-c)}=\frac{EP}{a-c}.$$

If DX, EY, FZ be perpendiculars on the tangent at P to the nine-points circle and PQ be the diameter, we see from similar triangles DPX and DPQ

$$PQ.DX=DP^2.$$

$$\therefore DX : EY : FZ :: DP^2 : EP^2 : FP^2 :: (b-c)^2 : (a-c)^2 : (a-b)^2 \quad (2)$$

We see also that both the radical axis and the tangent at P cut all the sides of the triangle DEF produced.

Therefore the radical axis and tho tangent at P are coincident on account of the relations (1) and (2).

Therefore the inscribed circle must be also touched at P by the radical axis.

Hence the in-circle and the nine-points circle touch at P.

R. P. PARANJPYE.

## The Isochord.

To determine a point J, such that the chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ drawn through it parallel to the sides of ABC shall be equal.

Let $\alpha\alpha'=x$: and let $(pqr)$ be the normal c's of J.

Then $\frac{x}{a}=\frac{h-p}{h}=1-\frac{ap}{2\Delta}$: ($h$=perp. from A on BC)

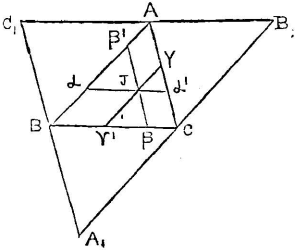
Therefore

$$x\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=3-\frac{ap+bq+cr}{2\Delta}=2$$

$$\therefore x=\frac{2\,abc}{bc+ca+ab}=\frac{4\Delta}{h_1+h_2+h_3}$$

Also $\frac{ap}{2\Delta}=1-\frac{x}{a}=\frac{-bc+ca+ab}{bc+ca+ab}$

$$\therefore ap \propto -\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$



Let $p,q,r$, be the normal $c's$ of J, referred to the triangle $A_1B_1C_1$, which is formed by drawing lines through A,B,C parallel to the opposite sides.

The diagram shows that

$$\frac{x}{a}=\frac{p_1}{h}=\frac{p_1 a}{2\triangle}$$

$$\therefore\ a^2p_1=b^2q_1=c^2r_1=2\triangle x.$$

Hence the barycentric $c's$ of J in $A_1B_1C_1$ are $(1/a,\ 1/b,\ 1/c)$ so that J in this triangle is isotomic to the incentre. The name 'isochord' is proposed for J.

W. Gallatly.

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## The Face of the Sky for October and November 1911. Sidereal time at 8 p.m.

| Date | | October H. | M. | S. | | November H. | M. | S. |
|---|---|---|---|---|---|---|---|---|
| 1 | ... | 20 | 36 | 49 | ... | 22 | 39 | 2 |
| 8 | ... | 21 | 4 | 25 | ... | 23 | 6 | 38 |
| 15 | ... | 21 | 32 | 1 | ... | 23 | 34 | 14 |
| 22 | ... | 21 | 59 | 37 | ... | 0 | 1 | 50 |
| 29 | ... | 22 | 27 | 12 | ... | 0 | 29 | 26 |

From this table the constellations visible during the evenings of October can be ascertained by a reference to their position as given in the star chart.

### Phases of the Moon.

| | | October | H. | M. | | | November | H. | M. | |
|---|---|---|---|---|---|---|---|---|---|---|
| Full Moon | ... | 8 | 9 | 41 | A.M. | ... | 6 | 9 | 18 | P.M. |
| Last Quarter | ... | 15 | 5 | 16 | ,, | ... | 13 | 0 | 50 | ,, |
| New Moon | ... | 22 | 9 | 39 | ,, | ... | 21 | 2 | 19 | A.M. |
| First Quarter | ... | 30 | 0 | 12 | P.M. | ... | 29 | 7 | 12 | ,, |

### Planets.

Mercury will shine in the Eastern sky during the first week of October. It is in conjunction with the moon on October 22 and November 22; with Jupiter on Nov. 8.

Venus is a morning star. Its position on Oct. 15 at 8 P.M. will be R. A. 10*h*. 58*m*. 18*s*. Dec. 2° 30′ 2″ N. and on Nov. 15, R. A. 12*h*. 22*m*. 49*m*. Dec. 1° 26′ 16.″S. It is the conjunction with the moon on Oct. 18 and Nov. 16. It attains its greatest elongation (West) 46° 45′ on 26 Oct. and greatest brilliancy on Oct. 22.

Mars—The position of the planet at 8 P.M. on Oct. 15 will be R. A 4*h*. 37*m*. 46*s*. Dec. 21° 20′ 48″N and on Nov. 15, R. A. 2*h*. 58*m*. 21*s*. Dec. 14° 6′ 40″N. It will be in conjunction with the moon on Oct. 12. It is at opposition to the sun on Nov. 25; these two months will be very good for the study of its surface markings. The diameter has increased from 4·91″ in March to 18·32″ in the middle of November.

Jupiter—The position of the planet on Oct. 15 at 8 P.M. will be R. A. 15*h*. 2*m*. 13*s*. Dec. 16° 21′ 5″S. On Nov. 16′ it rises immediately after sunrise and sets immediately after sunset. It is in conjunction with the moon on Oct. 23 and on Nov. 20.

Saturn—The best time to observe Saturn will be near the date of his opposition to the sun (Nov. 10) when he will rise at sunset and be visible all night. His position is some way to the S. W. of the Pleiades and W. of the Hyades. His position at 8 P.M. on Oct. 15 will be. R. A. 3*h*. 7*m*. 50*s*. Dec. 14° 56′ 54″N. and on Nov. 15 R. A. 2*h*, 58*m*, 21*s* Dec. 14° 6′ 40″N. It will be in conjunction with the moon on Oct. 10 and Nov. 7.

Uranus is in conjunction with the moon on Oct. 2, 30, and on Nov. 25.

Neptune is in conjunction with the moon on Oct. 15 and Nov. 12.

## Eclipses.

An annular eclipse of the sun partially visible in India will occur on the morning of Oct. 22. The line of the central eclipse passes from Turkestan through Tibet and South-east China into the China sea. It then passes between the Philippine Islands and Borneo and through the southern part of the Island of New Guinea. The phenomenon will be first visible at Srinagar in Kashmir where it will commence at sunrise. Kashmir and the Punjab will see the eclipse earlier than any other part of India. Next it will be visible in the United Provinces and Rajputana. At the 7th hour of the morning it will become visible from all parts of Upper India. Then gradually the line will move down bringing the Deccan inside the penumbral cone. When the eclipse becomes visible at Cape Comorin, we know that India is entirely inside the shadow. For over an hour it will remain completely within the penumbra. Sind and Punjab will first see the sun not eclipsed, next Rajputana and then the United Provinces and the Deccan, lastly Bengal and Assam, the last station where it will be visible in India being Dibrugarh.

Multan, Karachi, Haiderabad (Sind), Quetta, Peshwar, Attock and Rawal Pindi will see the sunrise more or less eclipsed. The eclipse com-

mences at Lahore ten minutes after sun rise. The annexed table gives the precise standard times of the beginning, middle and end of the phenmenon at some important cities in India.

| | | | | |
|---|---|---|---|---|
| Bombay | ... | 7. 9.36. | 8. 4.12. | 9. 3. 6. |
| Madras | ... | 7.22.18. | 8.17. 0. | 9.17. 0. |
| Lahore | ... | 6.51.34. | 7.57.45. | 9.11.47. |
| Allahabad | ... | 6.57. 3. | 8. 6. 8. | 9,23.50. |
| Nagpur | ... | 7. 3. 3. | 8. 6.54. | 9.18. 0. |
| Rangoon | ... | 8.13. 1. | 9.33.17. | 11. 3.15. |

There will be a penumbral eclipse of the moon visible in India on the 6th November 1911.

| | | H. | M. |
|---|---|---|---|
| First contact | ... | 7 | 9. |
| Last ,, | ... | 11 | 4. |

## Meteors.

| | R. A. | Dec. | Character. |
|---|---|---|---|
| October 2 | 230 | +52 | Slow, bright |
| ,, 4 | 310 | 79 | Slowish |
| ,, 8 | 77 | +31 | Swift streaks |
| ,, 8 to 14 | 45 | +58 | Small, short |
| ,, 14 | 133 | +68 | Rather swift |
| ,, 15 | 31 | + 9 | Slow |
| ,, 18 to 20 | 92 | +15 | Swift streaks |
| ,, 23 | 100 | +13 | ,, |
| ,, 29 | 109 | +23 | Very swift |
| November 1 | 43 | +22 | Slow bright |
| ,, 2 | 58 | + 9 | ,, |
| ,, 5—Dec 4 | 162 | +58 | Swift streaks |
| ,, 10—12 | 133 | +31 | Very swift streaks |
| ,, 14—16 (Leonids) | 150 | +22 | Swift streaks |
| ,, 16—28 | 154 | +41 | ,, |
| ,, 20—23 | 63 | +23 | Slow, bright |
| ,, 17—23 (Andromedids) | 25 | +43 | Very slow trains |
| ,, 25—Dec 12 | 189 | +73 | Rather swift |
| ,, 30 | 180 | +58 | Swift, streaks |

V. Ramesam.

# SOLUTIONS.

## Question 278.

(K. J. Sanjana, M.A.) :—When $m$ is an integer prove that $m(m+1)(11m^3+44m^2+71m+54)$ is a multiple of 20. Also, examine when the expression is a multiple of 80.

*Solution by M. Bhimasena Rao, M. K. Sadasiva Aiyar, B.A., B.C.E., N. Sankara Aiyar, B.A., and another.*

$$\begin{aligned} & m(m+1)(11m^3+44m^2+71m+54) \\ & \quad = m(m+1)(m+2)(11m^2+22m+27) \\ & \quad = m(m+1)(m+2)\{11(m-1)(m+3)+60\} \\ & \quad = 11(m-1)m(m+1)(m+2)(m+3)+60m(m+1)(m+2) \quad \ldots \text{ (1)} \end{aligned}$$

The first term is divisible by 5 ! or 120 and the second by $60\times 3!$ or 360. The given expression is therefore not only a multiple of 20 but of 120.

Again (1) is equal to

$$\begin{aligned} & 6(m-1)m(m+1)(m+2)(m+3)+40m(m+1)(m+2) \\ & \quad +5(m-1)m(m+1)(m+2)(m+3)+20m(m+1)(m+2) \\ & \quad = \text{a multiple of } 30+5\, m(m+1)(m+2)\{(m-1)(m+3)+4\} \\ & \quad = \text{a multiple of } 80+5\, m(m+1)^2(m+2). \end{aligned}$$

When the given expression is divisible by 80,

$$m(m+1)^2(m+2) \text{ is divisible by } 16$$

Let $m$ be odd, then $m+2$ is odd and $m+1$ is of the form $4n$ is $(m+1)^2$ is a multiple of 16, *i.e.*, $m=4n-1$.

Let $m$ be even, then $m+1$ is odd and $m+2$ is even. If $m\,(m+2)$ is a multiple of 16, $m$ or $m+2$ is a multiple of 8, *i.e.*. $m=8n$ or $8n-2$.

Hence when $m$ is of the form $4n-1$, $8n$ or $8n-2$, the given expression is divisible by 80 and also by 240.

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## Question 283.

(S. Ramanujam) :—Shew that is possible to solve the equations.

$$\begin{aligned} x+y+z&=a, & p^3x+q^3y+r^3z&=d, \\ px+qy+rz&=b, & p^4x+q^4y+r^5z&=e, \\ p^2x+q^2y+r^2z&=c, & p^5x+q^5y+r^5z&=f. \end{aligned}$$

where $x, y, z, p, q, r,$ are the unknowns. Solve the above when $a=2$, $b=3$ $c=4$, $d=6$, $e=12$ $f=32$.

*Solution by*: (1) *by M. K. Sadasiva Aiyar, B.A., B.C.E.*, (2) *by Sankara Aiyar B.A.*

(1) Solving the first three equations we have.

$$x\begin{vmatrix}1&1&1\\p&q&r\\p^2&q^2&r^2\end{vmatrix}=\begin{vmatrix}a&1&1\\b&q&r\\c&q^2&r^2\end{vmatrix}\quad \&c.$$

Similarly from the next three we get

$$px\begin{vmatrix}1&1&1\\p&q&r\\p^2&q^2&r^2\end{vmatrix}=\begin{vmatrix}b&1&1\\c&q&r\\d&q^2&r^2\end{vmatrix}$$

Similarly for $p^2x$ and $p^3x$.

Thus we have

$$p^3\begin{vmatrix}a&1&1\\b&q&r\\p^2&q^2&r^2\end{vmatrix}=p^2\begin{vmatrix}b&1&1\\c&q&r\\d&q^2&r^2\end{vmatrix}=p\begin{vmatrix}c&1&1\\d&q&r\\e&q^2&r^2\end{vmatrix}=\begin{vmatrix}d&1&1\\e&q&r\\f&q^2&r^2\end{vmatrix}$$

$$\therefore p=\frac{bqr-c\ (q+r)+d}{aqr-b(q+r)+c}=\frac{cqr-d\ (q+r)+e}{bqr-c\ (q+r)+d}=\frac{dqr-c(q+r)+f}{cqr-d(q+r)+c}$$

These two equation give $qr+q+r$; thence $q+r$ and also $px$ is found readily, as also $y+z$.

Particular case:

$$\frac{3\ qr-4\ (q+r)+6}{2qr-3\ (q+r)+4}=\frac{4\ qr+6\ (q+r)+12}{3\ qr-4\ (q+r)+6}=\frac{6\ qr-12(q+r)+32}{4\ qr-6\ (q+r)+12}$$

$$=p=\frac{5-(q+r)}{1}=\frac{2}{qr}=\frac{qr-6}{-(q+r)}$$

$$\therefore\ q^3r^3-6\ q^2r^2+10\ qr-4=0\ \therefore\ qr=2 \text{ or } 2\pm\sqrt{2}\quad q+r=4 \text{ or } 3\pm\sqrt{2}$$

$$\therefore\ q=2+\sqrt{2} \text{ or } 1$$

$$r=2\pm\sqrt{2} \text{ or } 2\pm\sqrt{2}$$

$$p=\quad 1 \text{ or } 2\pm\sqrt{2}$$

Substituting we get $y=\dfrac{3-2\sqrt{2}}{2\sqrt{2}}$; $z=\dfrac{\sqrt{2}+1}{2\sqrt{2}}$; $x=\dfrac{5\sqrt{2}-4}{2\sqrt{2}}$

(2) Taking the first four equations and eliminating $x$, $y$, $z$ we get

$$\begin{vmatrix}1&1&1&a\\p&q&r&b\\p^2&q^2&r^2&c\\p^3&q^3&r^3&d\end{vmatrix}=0$$

i e, on simplification

$$a\,pqr - b\Sigma pq + c\Sigma p - d = 0$$

and similarly

$$b\,pqr - c\Sigma pq + d\Sigma p - e = 0$$

$$c\,pqr - d\Sigma pq + e\Sigma p - f = 0$$

From these $pqr$, $\Sigma pq$, $\Sigma p$ may be evaluated and we get by solving these $p$, $q$ and $r$ and consequently $x$, $y$ and $z$.

In the particular case we have

$$\frac{pqr}{\begin{vmatrix} 3 & 4 & 6 \\ 4 & 6 & 12 \\ 6 & 12 & 32 \end{vmatrix}} = \frac{+\Sigma pq}{\begin{vmatrix} 2 & 4 & 6 \\ 3 & 6 & 12 \\ 4 & 12 & 32 \end{vmatrix}} = \frac{\Sigma p}{\begin{vmatrix} 2 & 3 & 6 \\ 3 & 4 & 12 \\ 4 & 6 & 32 \end{vmatrix}} = \frac{+1}{\begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 6 \\ 4 & 6 & 12 \end{vmatrix}}$$

i.e., $\frac{pqr}{-8} = \frac{\Sigma pq}{-24} = \frac{\Sigma p}{-20} = \frac{1}{-4}$

i.e. $pqr = +2,\ \Sigma pq = 6,\ \Sigma p = +5.$

Solving we get $p=1$, $q=2+\sqrt{2}$, $r=2-\sqrt{2}$ and two corresponding sets by interchanging these values substituting in the original equations we get $x$, $y$, $z$.

## Question 288.

(D. D. Kapadia, M.A., B.Sc.) :—Solve the equation—

$$\begin{vmatrix} a^2+x, & ba, & ac, & \ldots & \ldots \\ ab, & b^2+x, & bc, & \ldots & bl \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ al, & bl, & cl, & \ldots & l^2+x \end{vmatrix} = 0$$

*Solution by N. Sankara Aiyar, B.A. and S. Narayan, B.A., L.T., and another.*

The expansion of the determinant

$$\begin{vmatrix} x+a_{11}, & a_{12}, & a_{13}, & \ldots & a_{1n} \\ a_{21}, & x+a_{22}, & a_{23}, & \ldots & a_{2n} \\ a_{31}, & a_{32}, & x+a_{33}, & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & a_{n3}, & \ldots & x+a_{nn} \end{vmatrix}$$

is, $x^n + x^{n-1}\Sigma(a_{pp}) + x^{n-2}\Sigma(a_{pp}, a_{qq}) + x^{n-3}\Sigma(a_{pp}, a_{qq}. a_{rr}) + \dots$

where $(a_{pp}, a_{qq})$ stands for the determinant $\begin{vmatrix} a_{pp}, a_{pq} \\ a_{qp}, a_{qq} \end{vmatrix}$; and so on.

In the present case all the determinants of the second and higher order vanish. For $a_{pp}$, $a_{pq}$, $a_{pr}$....may be written in the form $a_p\, a_p$, $a_p.a_q$, $a_p.a_r$,...by putting $a = a_1$, $b = a_2$, $c = a_3$... So that

$$(a_{pp}, a_{qq}) = a_p\, a_q \begin{vmatrix} a_p, & a_q \\ a_p, & a_q \end{vmatrix} = 0 \text{; and so on.}$$

Hence the equation proposed reduces to

$$x^n + x^{n-1}\Sigma(a_p)^2 = 0$$

$$\therefore \quad x = 0, -\Sigma(a^2).$$

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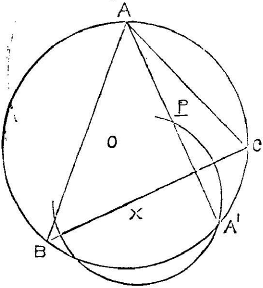
## Question 290.

(PROF. K. J. SANJANA) :—Construct a triangle, having given the circumcircle, the orthocentre, and that one side passes through a given point: and examine the number of solutions. (Instead of the orthocentre, either the centroid or the nine point centre may be given).

*Solution by R. Srinivasan M.A., D. Jagadisan, and R. D. Karve B.A.*

Let P to the orthocentre, $x$ the given point and O the circumcentre. With $x$ as centre and $x$P as radius describe a ⊙ cutting the given ⊙ in A′. Draw $x$D $\perp^r$ to PA′ meeting the circumcircle in B and C. Let PA′ meet the circumcircle in A. Then ABC is the triangle required.

The proof is evident.




If the new circle cuts the given circle in two points, we have two solutions, if it touches, one solution, if the circles do not cut all, then no solution is possible.

If the centroid or the nine-point centre be given, the orthocenter can be determined. Then the same method applies.

*Addition solutions by S. Narayan B.A., L.T., N. Sankara Aiyar, and G. Ramachandran.*

## Question 292.

(MUKANDLAL, M.A.) :—Forces $P\left(\tan\frac{A}{2}-\tan\frac{B}{2}\right)$, $P\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)$ and $P\left(\tan\frac{C}{2}-\tan\frac{A}{2}\right)$ act along the sides AB, BC, CA of a triangle ABC. Prove that their resultant is $2P\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$, and give a geometrical construction for the resultant.

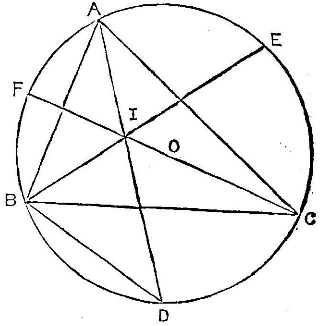
*Solution (1) by S. Purushotham, and (2) by R. Srinivasan, M.A.*

(1) Put $P=R$ (circumradius).

$R\tan\frac{1}{2}A$ along AB, and $R\tan\frac{1}{2}A$ along $CA=2R\sin\frac{1}{2}A$, along AI, where I is the incentre of ABC.

Let AI, BI, CI produced meet the circumcircle in D,E,F.

Then $ID=DB=2R\sin\frac{1}{2}A$.




Hence the given forces equal to forces ID, IE, IF, (*i.e.*) to 3I G, where G, is centroid of DEF. But I is the orthocentre of DEF.

$\therefore\ 3IG_1=2IO=2\sqrt{R^2-2Rr}$,

$=2R\sqrt{(1-8\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)}$.

The resultant passes through the incentre I,

$\because\ \Sigma P\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)r=0$

It passes also through the circumcentre O,

$$\therefore \Sigma P . R \cos A\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right)$$

$$=\frac{P . R}{2\,\Pi \cos\frac{A}{2}}\,\Sigma \cos A . 2 \sin\frac{B-C}{2} . \cos\frac{A}{2}$$

$$=\frac{P . R}{2\,\Pi \cos\frac{A}{2}} . \Sigma \cos A(\cos C-\cos B)=O.$$

The resultant equals the sum of the parts of the forces along OI. If $\theta$ be the angle between OI and BC,

$$\cos\theta=\frac{s-b-\frac{a}{2}}{OI}=\frac{c-b}{2 . OI}$$

$$\therefore \text{The resultant}=\Sigma P\left(\tan\frac{B}{2}-\tan\frac{C}{2}\right) . \frac{c-b}{2 . OI}$$

$$=\frac{P . R}{OI . \Pi \cos\frac{A}{2}}\,\Sigma \sin\frac{B-C}{2} . \cos\frac{A}{2} . (\sin C-\sin B)$$

$$=\frac{P.R.}{OI.\Pi \cos\frac{A}{2}}\sum 2 \sin\frac{B-C}{2}\cos\frac{A}{2} . \sin\frac{C-B}{2}\sin\frac{A}{2}$$

The absolute value of the resultant.

$$=\frac{P.R.}{2OI.\Pi \cos\frac{A}{2}}\sum \sin A\{1-\cos(B-C)\}$$

$$=\frac{P.R.}{2OI.\Pi \cos\frac{A}{2}}\sum\left\{\sin A-\tfrac{1}{2}(\sin 2B+\sin 2C)\right\}$$

$$=\frac{P.R.}{2OI.\Pi \cos\frac{A}{2}}\left\{4\,\Pi \cos\frac{A}{2}-4\,\Pi \sin A\right\}$$

$$\frac{2P.R}{OI} . \left(1-8\,\Pi \sin\frac{A}{2}\right)$$

$$=2 P . \frac{1-8\,\Pi \sin\frac{A}{2}}{\sqrt{1-8\Pi\sin\frac{A}{2}}}$$

$$=2 P\sqrt{1-8\,\Pi\left(\sin\frac{A}{2}\right)}$$

*Additional Solution by J. C. Swaminarayan, M.A.*

## Question 301.

(*D. D. Kapadia, M.A., B.S.C.*):—Prove that in the neighbourhood of a regular point of a plane curve, the equation to the curve referred to the tangent and normal as axes of $x$ and $y$ may be written in either of the forms,

$$y=\frac{x^2}{2\rho}-\frac{\rho^1}{6\rho^2}x^3+(3+2\rho^{1^2}-\rho\rho^{11})\frac{x^4}{24\rho^3}+\ldots$$

$$x=(2\rho y)^{\frac{1}{2}}+\tfrac{1}{3}\rho^1 y+\left(\frac{1}{12}\rho\rho^{11}-\tfrac{1}{4}-\frac{1}{36}\rho^{12}\right)\times\frac{2}{\rho}\Big)^{\frac{1}{2}}y^{\frac{3}{2}}$$

where $\rho\ \rho'\ \rho''$ denote the radius of curvature and its first and second rates of variation along the arc of the curve at the given point.

Hence find the locus of the centres of the conics having four point contact with the curve at the point and express the condition that the conic of the closest contact should be a parabola. (Math Tripos 1910).

*Solution by J. C. Swaminarayan, M.A., and N. Sankara Aiyar, B.A.*

By Maclaurin's theorem

$$y=(y)_o+x\left(\frac{dy}{dx}\right)_o+\frac{x^2}{\lfloor 2}\left(\frac{d^2y}{dx^2}\right)_o+\frac{x^3}{\lfloor 3}\left(\frac{d^3y}{dx^3}\right)_o+\frac{x^4}{\lfloor 4}\left(\frac{d^4y}{dx^4}\right)_o+\ldots$$

Now, $\dfrac{dy}{dx}=\tan\psi$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_o=0$

$$\frac{d^2y}{dx^2}=\sec^2\psi\frac{d\psi}{ds}\times\frac{ds}{dx}$$

$$=\sec^3\psi\times\frac{1}{\rho}=\frac{1}{\rho}\sec^3\psi$$

$$\left(\frac{d^2y}{dx^2}\right)_o=\frac{1}{\rho}$$

$$\frac{d^3y}{dx^3}=\frac{1}{\rho}\times 3\sec^2\psi\left(\sec\psi\tan\psi\right)\frac{d\psi}{ds}\times\frac{ds}{dx}-\frac{1}{\rho^2}\frac{d\rho}{ds}\times\frac{ds}{dx}\sec^3\psi$$

$\therefore$ $$\frac{d^3y}{dx^3}=\frac{3\sec^4\psi\tan\psi}{\rho^2}-\frac{1}{\rho^2}\frac{d\rho}{ds}\sec^4\psi$$

$\therefore$ $$\left(\frac{d^3y}{dx^3}\right)_o=-\frac{\rho'}{\rho^2}$$

Also $$\left(\frac{d^4y}{dx^4}\right)_o=\frac{3}{\rho^3}+\frac{2}{\rho^3}\left(\frac{d\rho}{ds}\right)^2-\frac{1}{\rho^2}\frac{d^2\rho}{ds^2}$$

$$=\frac{3+2\left(\frac{d\rho}{ds}\right)^2-\rho\frac{d^2\rho}{ds^2}}{\rho^3}$$

Hence

$$y=\frac{x^2}{2\rho}-\frac{\rho'}{6\rho^2}x^3+\left(3+2\rho'^2-\frac{x^4}{24\rho^3}\rho\rho''\right)+\dots \quad (1)$$

where $\rho'$ denotes $\frac{d\rho}{ds}$ and $\rho''$ denotes $\frac{d^2\rho}{ds^2}$.

For the value of $x$ in terms of $y$, take for a first approximation.

$$y=\frac{x^2}{2\rho} \text{ and we get } x=(2\rho y)^{\frac{1}{2}}$$

Let $x=(2\rho y)^{\frac{1}{2}}+\text{A}\,y$ Substituting in (i) we get thet the coefficient of $y^{\frac{3}{2}}$ vanishes snd hence $\frac{2\text{A}}{(2\rho)^{\frac{1}{2}}}-\frac{\rho\rho'(2\rho)^{\frac{1}{2}}}{3\rho^2}=0$

$$i.e.\ \text{A}=\frac{\rho^1}{3}\ \therefore\ x=(2\rho y)^{\frac{1}{2}}+\frac{\rho'}{3}y$$

Next, let $x=(2\rho y)^{\frac{1}{2}}+\frac{\rho'}{3}y+\text{B}y^{\frac{3}{2}}$.

Substituting in (i) the coefficient of $y^2$ vanishes, and hence

$$\frac{1}{6\rho}\left(\frac{\rho'^2}{3}+6\text{B}(2\rho)^{\frac{1}{2}}-2\rho'^2+8+2\rho'^2-\rho\rho''\right)=0$$

$$\therefore\quad 12\text{B}\left(\frac{\rho}{2}\right)^{\frac{1}{2}}=\rho\rho''-3-\frac{\rho'^2}{3}.$$

$$\therefore\quad \text{B}=\left(\frac{2}{\rho}\right)^{\frac{1}{2}}\left(\frac{\rho\rho''}{12}-\frac{1}{4}-\frac{\rho'^2}{36}\right)$$

$$\therefore\quad x=\left(2\rho y\right)^{\frac{1}{2}}+\frac{\rho'}{3}y+\left(\frac{\rho\rho''}{12}-\frac{1}{4}-\frac{\rho'^2}{36}\right)\left(\frac{2}{\rho}\right)^{\frac{1}{2}}y^{\frac{3}{2}}+\dots\dots\dots$$

The conics having four point contact with the curve at the point will be $2y=ax^2+2hxy+by^2$ in which the values of $f(x)$, $f'(x)$, $f''(x)$ and $f'''(x)$ are the same as for the curve.

Now expanding in terms $y$ we get

$$x=\frac{-hy\pm\sqrt{h^2y^2-(by^2+2y)a}}{a}$$

$$=\frac{-hy+\sqrt{2ay}\sqrt{1-\frac{ab-h^2}{2a}y}}{a}$$

$$=\left(\frac{2ay}{a}\right)^{\frac{1}{2}}-\frac{hy}{a}-y^{\frac{3}{2}}\frac{ab-h^2}{2\sqrt{2a}}+\dots$$

Hence $-\frac{h}{a}=\frac{1}{3}\rho'$ from the given expansion.

But the centres of the conics are given by

$$\frac{x}{-h}=\frac{y}{a}=\frac{1}{ab-h^2}$$

$$\therefore \qquad \frac{x}{y}=\frac{\rho'}{3} \text{ is the locus of centres.}$$

If this conic is a parabola $ab-h^2=0$, and we get

$$3\rho\rho''-\rho'^2-9=0$$

or

$$\frac{d}{ds}\left(\frac{\rho'}{\rho^{\frac{1}{3}}}\right)=\frac{+3}{\rho^{\frac{4}{3}}}$$

$$\frac{d^2}{ds^2}\left(\rho^{\frac{2}{3}}\right)=\frac{2}{\rho^{\frac{4}{3}}}$$

Integrating

$$\frac{\rho^{\frac{2}{3}}(c\rho^{\frac{2}{3}}+4)^{\frac{1}{2}}}{c}-\frac{4}{c^{\frac{3}{2}}}\log\left(c^{\frac{1}{2}}\rho^{\frac{1}{3}}+\sqrt{c\rho^{\frac{2}{3}}+4}\right)=s+c'$$

where $c$, $c'$ are arbitrary constants.

If we take $c=0$, this reduces to

$$\rho=3s+c'$$

As an example we may take the curve $s=ke^{3\psi}$ the simplest curve of this series, the general case being,

$$s=ke^{3\psi}-\frac{c'}{3} \quad \textit{i.e.}, \text{ the equiangular spiral.}$$

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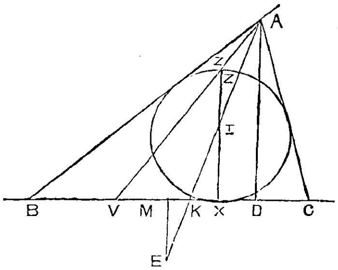
## Question 315.

(G. Ramachandran) :—Construct a triangle having given the incentre, the mid-point of the base, and the foot of the perpendicular from the vertex on the base.

*Solutions: (1) by K. J. Sanjana, M.A., and D. Jagadisan, B.A., (2) by M. V. Arunachala Sastry, M.A., (3) by S. Narayan B.A., L.T., and the Proposer.*

(1) Draw the diameter XIZ of the incircle perpendicular to BC; then it is known that AZ meets BC at V where the corrrsponding excircle touches it. Hence $BV=s-c=XC$; & $BM=MC$, $\therefore$ $VM=MX$, Therefore we have the following construction.—Draw IX perpendicular

**to MD & make IZ=IX, MV=MX; produce VZ to meet DA perpendicular to MD in A: then A is the vertex of the triangle.**




(2) Draw the circle through N and D to touch the incircle. This will be the N. P. circle whose radius is $\frac{1}{2}$ R. Hence R is known and therefore also SI, if S be the circumcentre. The rest of the construction is obvious.

(3) Suppose AI produced meets the circumcircle in E and the base BC in K; then we know that $EX . EA = EC^2 = EI^2$. Hence, also, $MX^2 = MK . MD$. Therefore, the construction is as follows:—Find the third proportional MK to MD, MX; produce KI to meet the $\perp^r$ through D in A, which will be the vertex of the triangle.

## QUESTIONS FOR SOLUTION.

**318.** (K. J. Sanjana, M.A.) :—Prove that

$$\int_0^{\pi/2} \log \sin\theta \,.\log\ \tan\theta\, d\theta = \frac{\pi^3}{16};$$

$$\int_0^{\pi/2} \log \sin\theta \,.\, \log\cos\theta\, d\theta = \frac{\pi}{2}\left\{(\log 2)^2 - \frac{\pi^2}{24}\right\};$$

$$\text{and} \int_0^{\pi/2} \theta \log \sin\theta d\theta = \frac{\pi}{4}\left\{-\frac{\pi}{2}\log 2 + \tfrac{1}{2}\,.\,\frac{1}{1^2} + \frac{1.3}{2.4}\cdot\frac{1}{2^2} + \frac{1.3\,5}{2.4.6}\cdot\frac{1}{3^2} + \ldots\right\}.$$

[Suggested by Quest. 308 of Mr. S. Ramanujan ; the first two integrals are given by *Wolstenholme.*]

**319.** (J. C. Swaminarayan, M.A.) :—

Determine a point X on a horizontal axis OX from which a particle can be projected with the least velocity so as to pass through two given points $(\xi, \eta)$ and $(\xi,^1\eta^1)$ in the same vertical plane as OX and shew that

$$OX = \frac{1}{2\rho}\ [(\xi+\xi^1)\rho + (\eta-\eta^1)\ (\xi-\xi^1) - (\xi-\xi^1)\ \sqrt{(\rho+\eta+\eta^1)^2 - 4\ \eta\eta^1}]$$

where $\rho = \{(\xi-\xi^1)^2 + (\eta-\eta^1)^2\}^{\frac{1}{2}}$

**320.** (N. Sankara Aiyar) :—Shew that the tangents to a system of conics inscribed in a given quadrilateral at the points where a fixed straight line meets them envelope a curve of the third class touching the given line and the sides of the given quadrilateral. [Math. Tripos. 1885.]

**321.** (S. Narayanan, B.A , L.T.) :—Shew that the join of the orthopoles of any two perpendicular straight lines bisects and is bisected by the join of their point of intersection with the orthocentre of the triangle.

**322.** (G. Ramachandran) :—AXYB is a semicircle described on AB as diameter. From X and Y perpendiculars XP and YQ are drawn to AB. A circle is drawn touching the semi circle externally and PX and QY produced. If OM be the perpendicular from O the centre of the circle on AB shew that $OM^2 = AQ.BP$.

**323.** (J. C. Swaminarayan, M.A.) :—Shew that with the usual notation of a plane triangle, the determinant

$$\begin{vmatrix} b^2 f^2 & , & c^2 f^2 & , & 1 \\ (ch+af)^2, & & c^2 g^2 & , & 1 \\ b^2 h^2 & , & (af+bg)^2, & & 1 \end{vmatrix}$$

$=2\ abcf\ (af+bg+ch)$

$+(gh+hf \cos C+fg \cos B-f^2 \cos A)+\ldots$

with two additional symmetrical results.

**324.** (R. Srinivasan M. A.) :—

Find the value of $\displaystyle\sum_{m=1}^{\infty} \frac{(-1)^m \cos mx}{(m+1)(m+2)}$.

**325.** (R. Srinivasan, M.A.) .—

If $a_n = \displaystyle\int_0^1 t\ (1-t)\ (2-t)\ldots\ldots\ldots\ldots(n-1-t)\ dt$, shew that

$$\sum_{0}^{\infty} (-1)^m\ a_m \frac{|x-1}{|x+m} = \log\left(1+\frac{1}{x}\right)$$

**326.** (N. Sankara Aiyar) :—Prove that

$$\frac{|n}{|n-p+1} - p \cdot \frac{|n-m}{|n-p-m+1} + \frac{p \cdot p-1}{1 \cdot 2} \frac{|n-2m}{|n-p-2m+1} - \ldots = 0,$$

if $n > p\ (m-1)$.

**327.** (S. Ramanujan) :—Shew that the Euler's constant viz

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots\ldots\ldots+\frac{1}{n}-\log n \text{ when } n \text{ is infinite}$$

is equal to

$$\log\ 2-1\left(\frac{2}{3^3-3}\right)-2\left(\frac{2}{6^3-6}+\frac{2}{9^3-9}+\frac{2}{12^3-12}\right)$$
$$-3\left(\frac{2}{15^3-15}+\frac{2}{18^3-18}+\cdots\frac{2}{39^3-39}\right)-\ldots\ldots,$$

the first term in the $n^{th}$ group being

$$\frac{2}{\left(\frac{3^n+3}{2}\right)^3-\left(\frac{3^n+3}{2}\right)}.$$

**328.** (M. BHIMASENA RAO) :—Show that the normal coordinates of the orthopole of $p\alpha+q\beta+r\gamma=0$ are $(-p+q\cos C+r\cos B)$ $(-p\cos B.\cos C+q\cos B+r\cos C)$ etc. and hence or otherwise that the envelope of a line whose orthopole lies on a given line is a parabola whose focus lies on the circumcircle of the triangle of reference.

**329.** (M. BHIMASENA RAO.) :—If the pedal circle of P with respect to a triangle ABC touches the nine points circle, shew that the sum of the angles PAB, PBC, PCA is constant.

**330.** (K. J. SANJANA.) :—Prove that

$$\text{(i) } 1.\frac{1}{1}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1.3}{2.4}\cdot\frac{1}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}+\ldots=\frac{\pi}{2};$$

$$\text{(ii) } 1.\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\cdot\frac{1}{5^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7^2}+\ldots=\frac{\pi}{2}\log 2$$

$$\text{and (iii) } 1.\frac{1}{1^3}+\frac{1}{2}\cdot\frac{1}{3^3}+\frac{1.3}{2.4}\cdot\frac{1}{5^3}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7^3}+\ldots=\frac{\pi}{4}\left\{(\log 2)^2+\frac{\pi^2}{12}\right\}$$

[*Note*—It is not easy to evaluate similar series with higher powers : I find

$$1.\frac{1}{1^4}+\frac{1}{2}\cdot\frac{1}{3^4}+\frac{1.3}{2.4}\cdot\frac{1}{5^4}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7^4}+\ldots$$

$$=\frac{\pi}{12}\log 2\left\{(\log 2)^2+\frac{\pi^2}{4}\right\}+\frac{5}{48}\sum\frac{1}{r^3}+\frac{1}{4}\sum\frac{1}{rst}.$$

Here $r$ has integral values from 1 to $\infty$; in the last term $r+s=t$ and $r+t\neq 2s$.]

**331.** (V. RAMASWAMI AIYAR, M.A.) :—Give a construction for the trisection of an angle by means of a *given* ellipse.

**332.** (V. RAMASWAMI AIYAR, M.A.) :—Four conics are circumscribed to a triangle having for directrices the lines joining the orthocentre to the in-and ex-centres of the triangle. Show that the foci of the conics, corresponding to these directrices, lie on the circumcircle and are the inverses of the Feuerbach points with respect to the polar circle of the triangle.